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# Piron's and Bell's Geometrical Lemmas

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The famous Gleason's Theorem gives a characterization of measures on lattices of subspaces of Hilbert spaces. The attempts to simplify its proof lead to geometrical lemmas that possess also easy proofs of some consequences of Gleason's Theorem. We contribute to these results by solving two open problems formulated by Chevalier, Dvurečenskij and Svozil. Besides, our use of orthoideals provides a unified approach to finite and infinite measures.

**KEY WORDS:** Gleason's Theorem; Bell inequalities; Bell's Geometrical Lemma; Weak Piron's Geometrical Lemma; Hilbert space.

## 1. INTRODUCTION

Let $H$ be a separable real, convex, or quaternion Hilbert space. The collection $L(H)$ of all its closed subspaces can be equipped with the partial order by inclusion (inducing bounds $\mathbf{0} = \{0\}, \mathbf{1} = H$) and *orthocomplementation* $P^{\perp} = \{x \in L(H) : x \perp P\}$. This way, $L(H)$ becomes an *orthomodular lattice* (see Svozil (1998) for a detailed exposition). Instead of closed subspaces we may equivalently work with the orthogonal projections onto these spaces. Gleason's Theorem (Gleason, 1957) characterizes measures on $L(H)$ for dim $H \geq 3$. The most difficult part of its proof deals with the case of a three-dimensional real vector space, $H = \mathbb{R}^3$. Attempts were made to simplify this part of the proof. They lead to Piron's and Bell's Geometrical Lemmas (Bell, 1964; Bell, 1966; Piron, 1976). These results enabled a simplification of the original proof of Gleason's Theorem (Cooke, 1985). Besides this, some important consequences (e.g., the nonexistence of two-valued measures on $L(\mathbb{R}^3)$ which is crucial for rejection of the hidden variables conjecture) can be obtained directly from these lemmas, without the need of proving the Gleason's Theorem in its full power.

Here we restrict attention to the principal case $H = \mathbb{R}^3$. We add another geometrical lemma (Lemma 3.4) and show its use in the problems studied in Chevalier *et al.* (2000). We refer to Dvurečenskij (1993) for a detailed historical

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introduction to the topic and to Chevalier *et al.* (2000) for motivation and basic notions used in this paper. We repeat here just the necessary minimum of basic terms.

## 2. BASIC NOTIONS

Let $S(\mathbb{R}^3)$ be the unit sphere in $\mathbb{R}^3$. We denote by $\angle(p, q)$ the angle of two vectors $p, q \in \mathbb{R}^3$. Throughout this paper, we fix a vector $p \in S(\mathbb{R}^3)$. It determines an orientation of $S(\mathbb{R}^3)$; for simplification we use the geographical terminology and call *p the north pole*. The *northern hemisphere* (relative to $p$) is the set $N_p$ of all $q \in S(\mathbb{R}^3)$ such that $\angle(p, q) \in (0, \frac{\pi}{2})$. The *equator* (relative to $p$) is the set $E_p$ of all unit vectors orthogonal to $p$. The *latitude* (relative to $p$) of a vector $q \in S(\mathbb{R}^3)$ is the angle $\omega_q$ between $q$ and the plane of the equator, i.e., $\omega_q = \frac{\pi}{2} - \angle(p, q)$.

A *great circle* in $S(\mathbb{R}^3)$ is the intersection of $S(\mathbb{R}^3)$ with a two-dimensional linear subspace. For each $q \in N_p$, there is a unique great circle, denoted by $C(q)$, which contains $q$ and intersects with the equator $E_p$ at the points orthogonal to $q$. Among the vectors of $C(q)$, $q$ has a maximal latitude.

The following tool was used:

**Weak Piron's Geometrical Lemma** (Piron, 1976) *Let $v$ and $q$ be two vectors in the northern hemisphere such that $v \in N_p$ lies below $C(q)$ (in the sense of latitude). Then there is a vector $s \in C(q)$ from the northern hemisphere such that $v \in C(s)$.*

To formulate further lemmas, we need the notion of measure. A mapping $m : L(\mathbb{R}^3) \to [-\infty, +\infty]$ is called a *measure* iff

- $m(\mathbf{0}) = 0$,
- $m(A + B) = m(A) + m(B)$ whenever $A, B$ are orthogonal subspaces of $\mathbb{R}^3$.

A measure is called *finite* if its range is finite, otherwise, it is called *infinite*. (An infinite measure attains exactly one of the improper values $\pm\infty$.) The *kernel* of a measure $m$ is

$$\ker m = \{P \in L(\mathbb{R}^3) : (\forall Q \in L(\mathbb{R}^3), Q \subseteq P : m(Q) = 0)\}.$$

If the range of a measure $m$ is nonnegative, then $\ker m = m^{-1}(0)$ and $m$ is called a *positive measure*; if, moreover, $m(\mathbf{1}) = 1$, it is called a *state*.

**Bell's Geometrical Lemma** (Bell, 1966) *Let $T$ and $U$ be one-dimensional subspaces of $\mathbb{R}^3$ and $m$ a state on $L(\mathbb{R}^3)$ such that $m(T) = 1$, $m(U) = 0$. Then $\angle(T, U) >$* arctan $\frac{1}{2}$.

The Bell's Geometrical Lemma can be strengthened as follows:

**Lemma 2.1.** *Let $T$ and $U$ be one-dimensional subspaces of $L(\mathbb{R}^3)$ and $m$ a state on $L(H)$ such that $m(T) = 1$, $m(U) = 0$. Then $T$,$U$ are orthogonal.*

We shall refer also to the following result for infinite measures (see also Dvurečenskij (1993)):

**Lemma 2.2.** (Lugovaja-Sherstnev (Lugovaja, 1980)) *Let* $m$: $L(\mathbb{R}^3) \to [-\infty, +\infty]$ *be an infinite measure and let* $U$ *and* $P$ *be subspaces of* $\mathbb{R}^3$ *of finite measure,* $\dim U = 1$, $\dim P = 2$. *Then* $U \subseteq P$.

Two of the open problems formulated in Chevalier et al. ((2000)) were:

1. Prove Lemma 2.1 using only Weak Piron's Geometrical Lemma.
2. Prove Lemma 2.2 using only Bell's Geometrical Lemma.

We do this using a new additional lemma. Moreover, we generalize the results so that they are applicable to both finite and infinite measures.

## 3. A GEOMETRICAL LEMMA

In this section, we add one simple geometrical lemma. Instead of measures, we formulate it for orthoideals.

*Definition 3.1.* An orthoideal in $L(\mathbb{R}^3)$ is a subset $I$ such that

- $A \leq B, B \in I \Longrightarrow A \in I$,
- $A, B \in I, A \perp B \Longrightarrow A \vee B \in I$.

Notice that an orthoideal need not contain the join of an arbitrary (nonorthogonal) pair of elements.

*Example 3.2.* If $m : L(\mathbb{R}^3) \to [-\infty, +\infty]$ is a measure, then $\ker m$ and $m^{-1}((-\infty, +\infty))$ are orthoideals.

Thus, orthoideals allow a unified approach to numerous results that were stated separately for kernels of finite measures and for preimages of $(-\infty, +\infty)$ under infinite measures, e.g., Bell's Geometrical Lemma may be formulated more generally as follows (and proved analogously):

**Lemma 3.3.** *Let* $I$ *be a proper orthoideal in* $L(\mathbb{R}^3)$ *containing subspaces* $U$,$P$ *with* $\dim U = 1$, $\dim P = 2$. *Then* $\angle(U, P^\perp) > \arctan \frac{1}{2}$.

For two-dimensional subspaces of $P$, $Q$ of $\mathbb{R}^3$, we define their angle, $\angle(P, Q) \in [0, \frac{\pi}{2}]$ as the (nonoriented) angle of their normals.

**Lemma 3.4.** *Let I be an orthoideal in $L(\mathbb{R}^3)$ containing two-dimensional subspaces P, Q with*

$$\angle(P, Q) = \beta \in (0, \arctan 2].$$

*Then there is a two-dimensional subspace $S \in I$ such that*

$$\angle(S, Q) \geq \beta + \arcsin\left(\frac{1}{4}\sin\beta\right).$$

*If $\beta = \arctan 2$, then S may be chosen orthogonal to Q.*

**Proof:** [2] Let us take a coordinate system $x$, $y$, $z$ such that the north pole is $p = (0, 0, 1)$, $P$ is in the plane $z = 0$ (the plane of the equator) and $Q$ has a unit normal vector $n_0 = (-\sin\beta, 0, \cos\beta)$. Thus $Q$ intersects the unit sphere in the great circle $C(q_0)$, where $q_0 = (x_0, 0, z_0)$, $z_0 = \sin\beta$, $x_0 = \cos\beta = \sqrt{1-z_0^2}$.

We construct a series $q_1, q_2, q_3, q_4$ of elements of the northern hemisphere as follows: For $i = 1, 2, 3, 4$, we choose $q_i \in C(q_{i-1})$ such that the orthogonal projections of $q_{i-1}$, $q_i$ to $P$ have an oriented angle $\frac{\pi}{4}$. Explicitly, for $i = 1$ we obtain

$$q_1 = \left(\frac{x_0}{z_0}z_1, \frac{x_0}{z_0}z_1, z_1\right) = \left(\frac{z_1}{z_0}\sqrt{1-z_0^2}, \frac{z_1}{z_0}\sqrt{1-z_0^2}, z_1\right),$$

where $z_1$ is determined from the normalizing condition $\|q_1\| = 1$:

$$z_1^2 = \frac{1}{\frac{2}{z_0^2}\left(1-z_0^2\right)+1} = \frac{z_0^2}{2-z_0^2}.$$

Using the rational function

$$g(t) = \frac{t}{2-t},$$

we may write $z_1^2 = g(z_0^2)$. Analogously, the $z$-coordinates $z_i$ of $q_i$, $i = 1, \ldots, 4$, satisfy $z_{i+1}^2 = g(z_i^2)$ and they may be expressed using the compositions (not powers!) $g^i$ of $g$ as

$$z_i^2 = g^i\left(z_0^2\right).$$

In particular,

$$g^2(t) = g(g(t)) = \frac{t}{4-3t},$$

$$g^4(t) = g^2(g^2(t)) = \frac{t}{16-15t},$$

[2] During the work on this paper, Maple V was successfully used for symbolic calculations.

$$z_4^2 = g^4\left(z_0^2\right) = \frac{z_0^2}{16 - 15z_0^2}.$$

Checking the orthogonal projections of $q_i$ to $P$, we see that $q_4$ is again in the plane $y = 0$, but its $x$-coordinate is negative:

$$q_4 = \left(-\sqrt{1 - z_4^2}, 0, z_4\right).$$

Due to our construction, $C(q_i), i = 0, \ldots, 4$, lie in two-dimensional subspaces belonging to $I$. In particular, the subspace $S \in I$ containing $C(q_4)$ has a unit normal vector

$$n_4 = \left(z_4, 0, \sqrt{1 - z_4^2}\right).$$

The assumption $\beta \leq \arctan 2$ ensures that $z_0^2 \leq \frac{4}{5}$. The restriction of $g$ to $[0, 1]$ is increasing and maps $[0, 1]$ into itself; the same holds also for $g^4$. Thus

$$z_4^2 = g^4\left(z_0^2\right) \leq g^4\left(\frac{4}{5}\right) = \frac{1}{5}$$

and equality holds iff $z_0^2 = \frac{4}{5}$. In particular,

$$z_0^2 + z_4^2 \leq 1$$

which means that $\angle(S, Q) = \angle(n_0, n_4)$ (not $\angle(n_0, -n_4)$ which is greater than $\frac{\pi}{2}$), and $S, Q$ are orthogonal iff $z_0^2 = \frac{4}{5}$, i.e., iff $\beta = \arctan 2$. Using the estimate

$$z_4^2 = \frac{z_0^2}{16 - 15z_0^2} \geq \frac{z_0^2}{16},$$

we obtain

$$z_4 \geq \frac{z_0}{4} = \frac{1}{4}\sin\beta, \angle(S, Q) = \angle(P, Q) + \angle(P, S) \geq \beta + \arcsin\left(\frac{1}{4}\sin\beta\right).$$

□

Using Lemma 3.4, we shall prove the following theorem:

**Theorem 3.5.** *Let $I$ be a proper orthoideal in $L(\mathbb{R}^3)$ containing subspaces $U$, $P$ with* $\dim U = 1$, $\dim P = 2$. *Then $U \subseteq P$.*

In view of Example 3.2, Theorem 3.5 is a common generalization of Lemmas 2.1, 2.2, and 3.3. We present two short proofs of Theorem 3.5, the first using Bell's Geometrical Lemma, the second using Weak Piron's Geometrical Lemma.

**Proof:** Using Bell's Geometrical Lemma: There is a one-dimensional subspace $V \subseteq P$ orthogonal to $U$. We shall apply Lemma 3.4 to prove that $Q = U \vee V \in I$ coincides with $P$. Let $\alpha$ be the supremum of all angles between two-dimensional subspaces from $I$. Suppose that $\alpha > 0$. If $\alpha \leq \arctan 2$, we can find an angle $\beta < \alpha$ arbitrarily close to $\alpha$ and two-dimensional subspaces in $I$ with angle $\beta$. Lemma 3.4 gives an angle $\beta + \arcsin(\frac{1}{4}\sin\beta)$ which is greater than $\alpha$ for a sufficiently large $\beta < \alpha$, a contradiction. The case $\alpha > \arctan 2$ is already excluded by the Bell's Geometrical Lemma 3.3 because $\angle(Q, P) = \angle(Q^{\perp}, P^{\perp}) = \frac{\pi}{2} - \angle(U, P^{\perp})$. The only remaining case is $\alpha = 0$. □

**Proof:** Using Weak Piron's Geometrical Lemma: As in the previous proof, we take the supremum $\alpha$ of all angles between two-dimensional subspaces from $I$. Let us assume that $\alpha > 0$. Lemma 3.4 excludes the case $\leq \arctan 2$. If $\alpha > \arctan 2$, we take two two-dimensional subspaces $P, Q_1 \in I$ with $\angle(P, Q_1) > \arctan 2$. Weak Piron's Geometrical Lemma induces that there is a two-dimensional subspace $Q \in I$ such that $\angle(P, Q) = \arctan 2$. The final statement of Lemma 3.4 gives $S \in I$ orthogonal to $Q$ which means that $I$ is not proper, a contradiction. So the only possible case is $\alpha = 0$. □

The most important corollary of the above results is the nonexistence of hidden variables in $L(\mathbb{R}^3)$; they correspond to *two-valued states*, i.e., states with range $\{0, 1\}$.

**Corollary 3.1.** *There is no two-valued state on* $L(\mathbb{R}^3)$.

This was the principal result of J. Bell (Bell, 1964, 1969). Alternative proofs were presented by Kochen-Specker-type theorems which find a finite sublattice of $L(\mathbb{R}^3)$ admitting no two-valued state; the first construction of this kind was given in (Kochen, 1967), nice overviews of recent simplifications can be found in (Pitowsky, 1998; Svozil, 1998).

## APPENDIX: THE RATIONAL SPACE

Let $\mathbb{Q}$ be the set of all rational numbers. We denote by $L(\mathbb{Q}^3)$ the set of all linear subspaces of $\mathbb{Q}^3$.

**Open problem:** Does $L(\mathbb{Q}^3)$ possess a two-valued measure?

In contrast to $L(\mathbb{R}^3)$, we cannot use Weak Piron's Geometrical Lemma to prove the nonexistence of such a measure:

**Proposition 3.1.** *Weak Piron's Geometrical Lemma does not hold for* $L(\mathbb{Q}^3)$.

**Proof:** We identify the north pole with $(0, 0, 1)$. The unit vectors $v = (\frac{4}{5}, 0, \frac{3}{5})$,

$q = (\frac{3}{5}, 0, \frac{4}{4})$ satisfy the assumptions of Weak Piron's Geometrical Lemma. Thus there is a vector $s = (x, y, z) \in \mathbb{R}^3$ satisfying the statement of Weak Piron's Geometrical Lemma. There are two such vectors differing only by the sign of the $y$-coordinate; without any loss of generality, we restrict attention to the case $y > 0$. We shall show that the one-dimensional subspace containing $s$ does not belong to $\mathbb{Q}^3$.

The elements of $C(q)$ are orthogonal to the normal vector $(-\frac{4}{5}, 0, \frac{3}{5})$. In particular, for $s \in C(q)$ we obtain

$$-\frac{4}{5}x + \frac{3}{5}z = 0. \tag{1}$$

The elements of $C(s)$ are orthogonal to the normal vector $n = (x_n, y_n, z_n)$; its coordinates are

$$z_n = \sqrt{x^2 + y^2},$$

$$x_n = \frac{-xz}{\sqrt{x^2 + y^2}},$$

$$y_n = \frac{-yz}{\sqrt{x^2 + y^2}}.$$

As $s$ is a unit vector, we may substitute

$$\sqrt{x^2 + y^2} := \sqrt{1 - z^2}.$$

The orthogonality of $p$ and $n$ implies that

$$\frac{4}{5}x_n + \frac{3}{5}z_n = 0,$$

i.e.,

$$\frac{4}{5}\frac{-xz}{\sqrt{1 - z^2}} + \frac{3}{5}\sqrt{1 - z^2} = 0. \tag{2}$$

We have two equations (1), (2) for variables $x$, $z$; they can be simplified to the system

$$-4x + 3z = 0, -4xz + 3(1 - z^2) = 0.$$

Its only positive solution is

$$z = \frac{1}{\sqrt{2}}, x = \frac{3}{4}z, y = \sqrt{1 - x^2 - z^2} = \sqrt{\frac{7}{32}},$$

so $\frac{y}{z} = \frac{\sqrt{7}}{4}$ is irrational. □

Thus, Weak Piron's Geometrical Lemma cannot be used to answer the problem.

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